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Almost self-duality and Harada rings

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Abstract

The existence of self-duality for left Harada rings was investigated by J. Kado and K. Oshiro [J. Algebra 211 (1999) 384–408]. Recently the author constructed examples of left Harada rings without self-duality [J. Algebra 241 (2001) 731–744]. In this paper, we investigate almost self-duality and rings of a certain class, which contains right co-Harada rings (equivalently left Harada rings). Here almost self-duality is a generalization of self-duality. The main purpose of the paper is to show that every ring of the class (particularly every right co-Harada ring) has an almost self-duality.

Keywords: Harada rings; PF rings; Morita duality; Almost self-duality

Introduction

In [9] Kado and Oshiro investigated the existence of self-duality for left Harada rings. On the other hand, in [10] the author constructed examples of left Harada rings without self-duality. In this paper, we shall prove that every right co-Harada ring has an almost self-duality, which is a generalization of self-duality.

Section 1 is devoted to study of several kinds of Morita dualities. In particular, we investigate rings with almost self-duality. By using the term of PF rings, we characterize such rings (Theorem 1.4).

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In Section 2 we study structures of some kind of rings (right PCH rings), which contains right co-Harada rings. Although most results are proved for right co-Harada rings by Oshiro [16–18], we shall restate the results for right PCH rings. As is similar to the case of right co-Harada rings and QF rings, by showing a relationship between right PCH rings and right PF rings, we determine the structure of right PCH rings (Theorem 2.17).

Finally, in Section 3 we prove that every right linearly compact right PCH ring (particularly right co-Harada ring) has an almost self-duality (Theorem 3.2). We also introduce the class of left PH rings, which contains left Harada rings. As well as the case of right co-Harada rings and left Harada rings, we prove that the class of right linearly compact right PCH rings coincides with that of left linearly compact left PH rings (Theorem 3.7).

Throughout this paper, all rings have identity, all modules are unitary and all homomorphisms are operated on the opposite side of scalars. Let A be a ring. We denote by $\mathrm{pi}(A)$ the set of all primitive idempotents of A. For a right A-module X, we denote the radical, the socle and the top of X (i.e., the factor module by its radical) by $J(X)$, $S(X)$, and $T(X)$, respectively. For a subset K of A (respectively Y of X), the symbol $l_X(K)$ (respectively $r_A(Y)$) denotes the annihilator of K in X (respectively Y in A). Similar notation will be used for left A-modules. We shall refer [21] for the results concerning Morita duality.

1. Dualities

In this section, we shall treat Morita dualities. In the first half, we recall the concepts of several kinds of Morita dualities and study basic properties of these dualities. In the later half, we define a ring A_e for a ring A and an idempotent e of A, which plays an important role in Section 2, and we investigate dualities of A_e.

We first recall that a ring A is *right Morita dual* to a ring B if there exists a bimodule ${}_BU_A$ that defines a Morita duality. In case A is right Morita dual to A itself, A has a *self-duality*. Following Simson [20], we say that a ring A is *right almost dual* to a ring B if there exist rings $A_1 = A, A_2, A_3, \dots, A_m, A_{m+1} = B$ such that each A_i is right Morita dual to A_{i+1}. In case A is right almost dual to A itself, A is said to have an *almost self-duality*. Clearly the concept of almost self-duality is a generalization of that of self-duality. It should be noted that the property for rings having an almost self-duality is Morita invariant as well as self-dualities.

Let ${}_BU_A$ be a bimodule that defines a Morita duality. Since $S(U_A) = S({}_BU)$, we simply denote this module by $S(U)$. Then there exist basic sets $\{e_1, e_2, \dots, e_n\}$ and $\{f_1, f_2, \dots, f_n\}$ of primitive idempotents for A and B, respectively, such that $S(f_iU) \cong T(e_iA)$ for each $i = 1, 2, \dots, n$. We note that $S(f_iU) \cong T(e_iA)$ is equivalent to $S(Ue_i) \cong T(Bf_i)$.

Let ${}_AU_A$ define a self-duality. We say that ${}_AU_A$ defines a *weakly symmetric self-duality* if $S(eU) \cong T(eA)$ for each $e \in \mathrm{pi}(A)$. As we noted above, the condition of weakly symmetric self-duality is left-right symmetric. A ring A is said to have a weakly symmetric self-duality in case there exists a bimodule ${}_AU_A$ that defines a weakly symmetric self-duality. (See [11, p. 12].)

We first note the following two lemmas:

Lemma 1.1 [12, Corollary 3.4]. *Let ${}_BU_A$ define a Morita duality and let $e \in A$ and $f \in B$ be idempotents with orthogonal decompositions $e = \sum_{i=1}^n e_i$ and $f = \sum_{i=1}^n f_i$ of primitive idempotents such that $S(f_iU) \cong T(e_iA)$ for each $i = 1, 2, \ldots, n$. Then the bimodule ${}_{fBf}fUe_{eAe}$ defines a Morita duality.*

Lemma 1.2. *Let A be a ring and let e be a non-zero idempotent of A.*

(1) *If A has a right Morita duality, then so does eAe.*
(2) *If A has an almost self-duality, then so does eAe.*
(3) *If A has a weakly symmetric self-duality, then so does eAe.*

Proof. (1) This is by Lemma 1.1.

(2) Since A has an almost self-duality, there exist bimodules ${}_{A_2}U_{1A_1}$, ${}_{A_3}U_{2A_2}, \ldots, {}_{A_{m+1}}U_{mA_m}$ that define a Morita duality, where $A_1 = A_{m+1} = A$. For each $i = 1, 2, \ldots, m$, there exists a basic set $\{e_{i1}, e_{i2}, \ldots, e_{in}\}$ of primitive idempotents for A_i such that $S(e_{i+1,j}U_i) \cong T(e_{ij}A_i)$ for $1 \leqslant i \leqslant m-1$ and $1 \leqslant j \leqslant n$. Then for ${}_{A_{m+1}}U_{mA_m} = {}_{A_1}U_{mA_m}$, there exists a permutation σ of $\{1, 2, \ldots, n\}$ such that $S(e_{1,\sigma(j)}U_m) \cong T(e_{mj}A_m)$ for $1 \leqslant j \leqslant n$. For every non-empty subset K of $\{1, 2, \ldots, n\}$, let $e_K = \sum_{k\in K} e_{1k}$. Then it follows from Lemma 1.1 that e_KAe_K is right almost dual to $e_{\sigma(K)}Ae_{\sigma(K)}$.

Now we may assume that $e = e_K$ for some subset K of $\{1, 2, \ldots, n\}$ because the existence of almost self-duality is Morita invariant. Then $eAe = e_KAe_K$ is right almost dual to $e_{\sigma^i(K)}Ae_{\sigma^i(K)}$ for all positive integer i. Since the order of σ is finite, there exists a positive integer i such that $K = \sigma^i(K)$. Therefore e_KAe_K is right almost dual to e_KAe_K, that is, eAe has an almost self-duality.

(3) This is by [11, Lemma 2.10(2)]. □

As the following example shows, a similar statement of the lemma above does not hold for self-dualities.

Example 1.3. By using the results of [4,19], Kraemer constructed pairwise non-isomorphic five artinian rings $A_1, A_2, \ldots, A_5$ such that each A_i is right Morita dual to A_{i+1} for $1 \leqslant i \leqslant 4$ and A_5 is right Morita dual to A_1 [11, Proposition 6.5(1) and Remark 6.1]. Then each A_i does not have a self-duality but has an almost self-duality. Let $B = A_1 \times A_2 \times \cdots \times A_5$ be the product ring and let $e_1 = (1, 0, \dots, 0)$ be the central idempotent of B. Then B has a self-duality and $e_1 B e_1 \cong A_1$ does not have a self-duality.

We can now characterize rings with almost self-duality. To state this, we recall that a ring A is a *right PF ring* if A_A is an injective cogenerator for the category of right A-modules. *Left PF rings* are defined similarly. A right and left PF ring is called a *PF ring*.

Theorem 1.4. *A ring* (*respectively artinian ring*) *A has an almost self-duality if and only if there exist a PF ring* (*respectively QF ring*) *R and an idempotent e of R such that $A \cong eRe$.*

Proof. ($\Leftarrow$). This is clear from Lemma 1.2(2).

($\Rightarrow$). Let A be a ring with almost self-duality and let $A_1 = A$. Then there exist bimodules ${}_{A_2}U_{1A_1}, {}_{A_3}U_{2A_2}, \dots, {}_{A_1}U_{mA_m}$ that define a Morita duality. Let $B = A_1 \times A_2 \times \cdots \times A_m$ be the ring product and let $V = U_1 \oplus U_2 \oplus \cdots \oplus U_m$ be the direct sum of additive groups. Clearly V becomes a (B, B)-bimodule that defines a Morita duality. By [21, Theorem 10.7] the trivial extension $R = B \ltimes V$ of B by V is a PF ring. Let $e \in R$ be the idempotent corresponding to $(1, 0, \dots, 0) \in B$. Then we have $A \cong eRe$. In case A is artinian, each A_i is artinian and each U_i is finitely generated on both left and right sides. Thus R is a QF ring. □

Remark 1.5. As we used in the proof above, it is well known that if ${}_AU_A$ defines a self-duality, then the trivial extension $R = A \ltimes U$ is a PF ring and A is isomorphic to a factor ring of R. We can regard Theorem 1.4 as an analogy of this fact.

As a weakly symmetric self-duality version of Theorem 1.4, we have the next theorem.

Theorem 1.6. *A ring* (*respectively artinian ring*) *A has a weakly symmetric self-duality if and only if there exist a PF ring* (*respectively QF ring*) *R with weakly symmetric self-duality and an idempotent e of R such that $A \cong eRe$.*

Proof. ($\Leftarrow$) This is clear from Lemma 1.2(3).

($\Rightarrow$) Let ${}_AU_A$ be a bimodule that defines a weakly symmetric self-duality. Let $B = A \times A$ be the product ring and let $V = U \oplus U$ be the direct sum of additive groups. Then V becomes a (B, B)-bimodule defined by

$$(u_1, u_2)(a_1, a_2) = (u_1 a_1, u_2 a_2) \quad \text{and} \quad (a_1, a_2)(u_1, u_2) = (a_2 u_1, a_1 u_2)$$

for all $a_1, a_2 \in A$ and $u_1, u_2 \in U$. As is similar to the proof of Theorem 1.4, V defines a self-duality and the trivial extension $R = B \ltimes V$ is a PF ring. By [10, Proposition 2.1 and Remark 2.2] R has a weakly symmetric self-duality. Let $e \in R$ be the idempotent corresponding to $(1, 0) \in B$. Then we have $A \cong eRe$. □

Let ${}_BU_A$ define a Morita duality. Then the annihilator correspondences $l_U : \mathcal{L}({}_AA_A) \to \mathcal{L}({}_BU_A)$ and $l_B : \mathcal{L}({}_BU_A) \to \mathcal{L}({}_BB_B)$ are lattice anti-isomorphisms, where $\mathcal{L}({}_AA_A)$ and $\mathcal{L}({}_BB_B)$ denote the lattices of ideals and $\mathcal{L}({}_BU_A)$ denotes the lattice of subbimodules. Composing these two isomorphisms, we have a lattice isomorphism $l_Bl_U : \mathcal{L}({}_AA_A) \to \mathcal{L}({}_BB_B)$. For these isomorphisms, the following two lemmas hold.

Lemma 1.7 [21, Corollary 2.5]. *Let ${}_BU_A$ define a Marita duality. For every proper ideal K of A, let $V = l_U(K)$ and $L = l_B(V)$. Then the bimodule ${}_{B/L}V_{A/K}$ defines a Marita duality.*

Lemma 1.8. *Let ${}_BU_A$ define a Marita duality. Then*

(1) $l_Bl_U(KL) = l_Bl_U(K) \cdot l_Bl_U(L)$ *for all ideals K and L of A.*
(2) $l_Bl_U(S(A_A)) = S(B_B)$.

Furthermore, let $e \in A$ and $f \in B$ be idempotents with orthogonal decompositions $e = \sum_{i=1}^n e_i$ and $f = \sum_{i=1}^n f_i$ of primitive idempotents such that $S(f_iU_A) \cong T(e_iA_A)$ for each $i = 1, 2, \ldots, n$. Then

(3) $l_Bl_U(AeA) = BfB$.

Proof. (1) This is by [13, Theorem 19.52(1)].

(2) Since B is semilocal (i.e., $B/J(B)$ is semisimple), we have $J({}_BU) = J(B)U$ and $S(B_B) = l_B(J(B))$. Thus it follows from the faithfulness of ${}_BU$ that

$$S(B_B) = l_B\bigl(J(B)\bigr) = l_B\bigl(J(B)U\bigr) = l_B\bigl(J({}_BU)\bigr).$$

Therefore by the double annihilator properties of ${}_BU_A$, we have

$$r_U\bigl(S(B_B)\bigr) = r_Ul_B\bigl(J({}_BU)\bigr) = J({}_BU) = l_U\bigl(S(A_A)\bigr)$$

and hence

$$l_Bl_U\bigl(S(A_A)\bigr) = l_Br_U\bigl(S(B_B)\bigr) = S(B_B).$$

(3) We show that $l_U(AeA) = r_U(BfB)$. If $l_U(AeA) \nleqslant r_U(BfB)$, since $S(fU)$ is essential in fU, there exists $u \in l_U(AeA)$ such that $0 \neq fu \in S(fU)$ and hence $0 \neq f_iu \in S(f_iU)$ for some i. By $S(f_iU) \cong T(e_iA)$ we have $f_iue_i \neq 0$, a contradiction. Therefore $l_U(AeA) \leqslant r_U(BfB)$. Similarly $r_U(BfB) \leqslant l_U(AeA)$ and hence $l_U(AeA) = r_U(BfB)$. $\square$

Using the lemmas above, we have the next lemma.

Lemma 1.9. *Let A be a ring and let K be a proper ideal of A. Then*

(1) *If A has a right Marita duality, then A/K has a right Marita duality.*
(2) *If A has an almost self-duality and $K = AeS(A_A)$ for some idempotent e of A, then A/K has an almost self-duality.*
(3) *If A has a weakly symmetric self-duality and $K = AeS(A_A)$ for some idempotent e of A, then A/K has a weakly symmetric self-duality.*

Proof. (1) This is by Lemma 1.7.

(2) Let A be a ring with almost self-duality. Then there exist rings $A_1 = A, A_2, \ldots, A_{m+1} = A$ such that each A_i is right Morita dual to A_{i+1}. Composing the lattice isomorphisms $\mathcal{L}({}_{A_i}A_{i\,A_i}) \to \mathcal{L}({}_{A_{i+1}}A_{i+1\,A_{i+1}})$ of annihilators, we obtain a lattice automorphism Θ of $\mathcal{L}({}_AA_A)$. Then by Lemma 1.7 A/L is right almost dual to $A/\Theta(L)$ for every proper ideal L of A. Therefore if $\Theta^n(L) = L$ for some positive integer n, then A/L has an almost self-duality. It follows from Lemma 1.8(3) and a similar way of the proof of Lemma 1.2(2) that for any idempotent e of A there is a positive integer n such that $\Theta^n(AeA) = AeA$. Hence by Lemma 1.8 we have

$$\Theta^n\bigl(AeS(A_A)\bigr) = \Theta^n(AeA)\Theta^n\bigl(S(A_A)\bigr) = AeAS(A_A) = AeS(A_A).$$

Therefore $A/AeS(A_A)$ has an almost self-duality.

(3) Let ${}_AU_A$ define a weakly symmetric self-duality and assume $K = AeS(A_A)$ for some idempotent e of A. Let $\overline{A} = A/K$ and let $V = l_U(K)$. Then by Lemmas 1.7 and 1.8 V is an $(\overline{A}, \overline{A})$-bimodule that defines a Morita duality. It is routine to see that $S(fV_{\overline{A}}) = T(f\overline{A}_{\overline{A}})$ for each $f \in \mathrm{pi}(\overline{A})$. Therefore ${}_{\overline{A}}V_{\overline{A}}$ defines a weakly symmetric self-duality. □

Remark 1.10. For self-dualities, a similar statement of Lemma 1.9(1) does not hold. By Example 1.3 the ring B has a self-duality and e_1Be_1 does not have a self-duality. Therefore, since e_1 is central, the factor ring $B/B(1 - e_1) \cong e_1Be_1$ does not have a self-duality.

In the rest of this section, we define a ring A_e for a ring A and an idempotent e of A and we investigate relationships of dualities between A and A_e. The ring A_e will play an important role in descriptions of structures of right co-Harada rings (right PCH rings) in Section 2.

Let ${}_BU_A$ define a Morita duality and let K and L be ideals of A. Set $V = l_U(K)$ and $W = l_U(L)$, the subbimodules of ${}_BU_A$, and set $M = l_B(V)$ and $N = l_B(W)$, the ideals of B. We define

$$\widetilde{A} = \begin{pmatrix} A & L \\ K & A \end{pmatrix}, \qquad \widetilde{B} = \begin{pmatrix} B & N \\ M & B \end{pmatrix} \quad \text{and} \quad \widetilde{U} = \begin{pmatrix} U & U/V \\ U/W & U \end{pmatrix}.$$

Then $\widetilde{A}$ and $\widetilde{B}$ are generalized matrix rings and $\widetilde{U}$ is a $(\widetilde{B}, \widetilde{A})$-bimodule.

Lemma 1.11. *With the setting as above, the $(\widetilde{B}, \widetilde{A})$-bimodule $\widetilde{U}$ defines a Marita duality.*

Proof. We regard $\widetilde{A}$ as a ring extension of the product ring $A \times A$. By [21, Theorem 4.5] A and U are linearly compact right A-modules. (For the definition and basic properties of linearly compact modules, see also [21].) Therefore, it follows from [21, Proposition 3.3 and Lemma 4.4] and the definitions of $\widetilde{A}$ and $\widetilde{U}$ that $\widetilde{A}$ and $\widetilde{U}$ are linearly compact as right $\widetilde{A}$-modules. It is easy to see that $\widetilde{U} \cong \mathrm{Hom}_{A\times A}(\widetilde{A}, U \oplus U)$ and $\widetilde{B} \cong \mathrm{End}_{\widetilde{A}}(\widetilde{U})$. Hence by [21, Proposition 7.3] ${}_{\widetilde{B}}\widetilde{U}_{\widetilde{A}}$ defines a Morita duality. □

Example 1.12. For a bimodule ${}_BU_A$ defining a Morita duality, let

$$A_n = \begin{pmatrix} A & A & \cdots & A & A \\ J(A) & A & \cdots & A & A \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ J(A) & J(A) & \cdots & A & A \\ J(A) & J(A) & \cdots & J(A) & A \end{pmatrix}$$

and

$$B_n = \begin{pmatrix} B & B & \cdots & B & B \\ J(B) & B & \cdots & B & B \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ J(B) & J(B) & \cdots & B & B \\ J(B) & J(B) & \cdots & J(B) & B \end{pmatrix}$$

be the $n \times n$ generalized matrix rings and let

$$U_n = \begin{pmatrix} U & U/S(U) & \cdots & U/S(U) & U/S(U) \\ U & U & \cdots & U/S(U) & U/S(U) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ U & U & \cdots & U & U/S(U) \\ U & U & \cdots & U & U \end{pmatrix}$$

be the module of $n \times n$ matrices. Since $S(U) = l_U(J(A)) = r_U(J(B))$, it follows from Lemmas 1.11 and 1.1 that U_n are (B_n, A_n)-bimodules that define a Morita duality.

For any ring A and an idempotent e of A, we define a generalized matrix ring by

$$A_e = \begin{pmatrix} A & Ae \\ eJ(A) & eAe \end{pmatrix}.$$

Suppose that ${}_BU_A$ defines a Morita duality and that e has an orthogonal decomposition $e = \sum_{i=1}^{n} e_i$ of primitive idempotents of A such that $e_iA \ncong e_jA$ if $i \neq j$. Then there exists an idempotent f of B with an orthogonal decomposition

$f = \sum_{i=1}^{n} f_i$ primitive idempotents of B such that $S(f_i U) \cong T(e_i A)$ for each i. Furthermore, we define

$$B_f = \begin{pmatrix} B & Bf \\ fJ(B) & fBf \end{pmatrix} \quad \text{and} \quad U_{f,e} = \begin{pmatrix} U & Ue/S(U)e \\ fU & fUe \end{pmatrix}.$$

Lemma 1.13. *With the setting as above, $U_{f,e}$ becomes a (B_f, A_e)-bimodule that defines a Morita duality.*

Proof. Let

$$\widetilde{A} = \begin{pmatrix} A & A \\ J(A) & A \end{pmatrix}, \qquad \widetilde{B} = \begin{pmatrix} B & B \\ J(B) & B \end{pmatrix}, \quad \text{and}$$
$$\widetilde{U} = \begin{pmatrix} U & U/S(U) \\ U & U \end{pmatrix}.$$

By Example 1.12 $\widetilde{U}$ is a $(\widetilde{B}, \widetilde{A})$-bimodule that defines a Morita duality. We define idempotents by

$$\tilde{e} = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} \in \widetilde{A} \quad \text{and} \quad \tilde{f} = \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix} \in \widetilde{B}.$$

It follows from Lemma 1.1 that the $(\tilde{f}\widetilde{B}\tilde{f}, \tilde{e}\widetilde{A}\tilde{e})$-bimodule $\tilde{f}\widetilde{U}\tilde{e}$ defines a Morita duality. This proves the statement of the lemma. □

For relationships of dualities between A and A_e, we have

Proposition 1.14. *For a ring A and an idempotent e of A,*

(1) *A has a right Morita duality if and only if so does A_e.*
(2) *A has an almost self-duality if and only if so does A_e.*
(3) *A has a weakly symmetric self-duality if and only if so does A_e.*

Proof. We note that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A_e \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cong A.$$

Therefore (1) follows from Lemmas 1.2(1) and 1.13.

(2) The implication ($\Leftarrow$) follows from the note above and Lemma 1.2(2). The implication ($\Rightarrow$) follows from Lemma 1.13 and a similar argument of the proof of Lemma 1.2(2).

(3) ($\Leftarrow$) This follows from the note above and Lemma 1.2(3).

($\Rightarrow$) Let ${}_AU_A$ define a weakly symmetric self-duality and let e be a non-zero idempotent of A. Let $\widetilde{A} = A_e$ and $\widetilde{U} = U_{e,e}$. By Lemma 1.13 the $(\widetilde{A}, \widetilde{A})$-bimodule

$\widetilde{U}$ defines a self-duality. Then it is routine to check that $S(\tilde{f}\widetilde{U}) \cong T(\tilde{f}\widetilde{A})$ and $S(\hat{g}\widetilde{U}) \cong T(\hat{g}\widetilde{A})$ for each $f \in \mathrm{pi}(A)$ and $g \in \mathrm{pi}(eAe)$, where

$$\tilde{f} = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{g} = \begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix}.$$

This shows that $\widetilde{U}$ defines a weakly symmetric self-duality. □

Remark 1.15. For self-dualities, the implication (⇒) of a similar statement of Proposition 1.14 does not hold. Indeed, as we noted in [10, Examples 3.1 and 3.2], there exist QF rings A and idempotents e of A such that A_e do not have a self-duality. (See also Example 3.3.)

2. Structure of co-Harada rings and PCH rings

In this section we show that basic results about structures of co-Harada rings in [16–18] still hold on more general class of rings. Most results and proofs are essentially due to Oshiro [16–18].

Lemma 2.1 (cf. [16, Lemma 3.3]). *Let A be a basic semiperfect ring, let $e \in \mathrm{pi}(A)$ such that eA is quasi-injective with simple essential socle, and let $f \in \mathrm{pi}(A)$ with $S(eA) \cong T(fA)$. Then*

(1) ${}_{eAe}eAf_{fAf}$ *has simple essential socle on both sides.*
(2) $S(eA) = S(eA_A)f = S({}_{eAe}eAf) = S(eAf_{fAf})$.

Proof. This lemma follows from [8, Lemmas 3.5, 2.1, 3.1]. □

Lemma 2.2 (cf. [16, Lemma 3.4]). *Let A be a basic semiperfect ring, let $e \in \mathrm{pi}(A)$ such that eA_A is injective with simple essential socle, and let $f \in \mathrm{pi}(A)$ with $S(eA) \cong T(fA)$. Let $e_1 = e, e_2, \ldots, e_n \in \mathrm{pi}(A)$ be orthogonal such that*

(a) $e_iA \cong J(e_{i-1}A)$ *for each* $i = 2, \ldots, n$,
(b) *if* $\mathrm{Hom}_A(T(fA), gA) \neq 0$ *for* $g \in \mathrm{pi}(A)$, *then* $gA \cong e_iA$ *for some* i.

Furthermore, let $g \in \mathrm{pi}(A)$ such that gA has simple essential socle. Then for any non-zero homomorphism $a : fA \to gA$, we have

(1) *If* $\mathrm{Im}(\alpha) \neq S(gA)$, *then for each* $k = 1, 2, \ldots, n$, *there exists a homomorphism* $\beta : gA \to e_kA$ *such that* $\mathrm{Im}(\beta\alpha) = S(e_kA)$.
(2) *If* $\mathrm{Im}(\alpha) = S(gA)$ *and* $g = e_t$ *for some* $1 \leqslant t \leqslant n$, *then for each* $k = l, 2, \ldots, t$, *there exists a homomorphism* $\beta : gA \to e_kA$ *such that* $\mathrm{Im}(\beta\alpha) = S(e_kA)$.

Proof. Under our assumptions, the proof of [16, Lemma 3.4] is available. □

We recall that a ring A is a *right QF-2 ring* if every indecomposable projective right A-module has simple essential socle. (See [7].) We also recall from [1, Proposition 10.7] that a module X is *finitely cogenerated* if and only if X has finitely generated essential socle. For all non-negative integer k, we denote the kth radical and the kth socle of a right A-module X by $J_k(X)$ and $S_k(X)$, respectively. If the ring A is semilocal, then $J_k(X) = XJ(A)^k$ and $S_k(X) = l_X(J(A)^k)$ for all non-negative integer k.

Lemma 2.3 (cf. [16, Proposition 3.5(1)]). *Let A be a basic semiperfect right QF-*2 *ring and let $e_1 = e, e_2, \dots, e_n$, $f \in \mathrm{pi}(A)$ satisfying the conditions* (a) *and* (b) *of Lemma* 2.2. *Then for each $k = 1, 2, \dots, n$,*

$$S_k({}_A Af) = S(e_1 A_A) + \cdots + S(e_k A_A).$$

Proof. Under our assumptions, the proof of [16, Proposition 3.5(1)] is available. □

Harada [6] studied the following two conditions:

- $(*)$ Every non-small left module contains a non-zero injective submodule.
- $(*)^*$ Every non-cosmall right module contains a nonzero projective direct summand.

Here, we call a module X *non-small* if X is not small in an injective hull of X and otherwise we call X *small*. Dually, we call X *non-cosmall* if there exists an epimorphism $\alpha : P \to X$ with P projective such that $\mathrm{Ker}(\alpha)$ is not essential in P and otherwise X *cosmall*. Following Oshiro [15], we say that a left artinian ring with $(*)$ is a *left Harada ring* and a ring with $(*)^*$ satisfying ACC on right annihilators is a *right co-Harada ring*. Oshiro [16] proved that the class of left Harada rings coincides with that of right co-Harada rings. We shall extend this result in Theorem 3.7. In this paper, we mainly deal with the class of semiperfect rings with $(*)^*$, which contains right co-Harada rings.

Some characterizations of semiperfect rings with $(*)^*$ are given in the next lemma. To state this, we say that a module X is *cohopfian* if every monomorphism $X \to X$ is an isomorphism. (See [13, Exercise 1.6].) It is clear that every indecomposable quasi-injective module is co-hopfian. Thus, in the condition (4) of the lemma below, fA is co-hopfian for each $f \in \mathrm{pi}(A)$.

Lemma 2.4. *For a semiperfect ring A, the following statements are equivalent.*

(1) *A satisfies $(*)^*$.*

(2) *There exists a basic set* $\{e_i\} \cup \{f_j\}$ *of primitive idempotents for* A *satisfying*
 (i) *each* $e_i A$ *is injective and each* $f_j A$ *is small,*
 (ii) *for each* $e_i A$*, there exists a non-negative integer* t_i *such that all* $J_0(e_i A), J_1(e_i A), \dots, J_{t_i}(e_i A)$ *are projective and* $J_{t_i+1}(e_i A)$ *is singular,*
 (iii) *each* $f_j A$ *can be embedded in* $e_i A$ *for some* i.

(3) *There exists a basic set* $\{e_{ij} \mid 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n(i)\}$ *of primitive idempotents for* A *satisfying*
 (i) $e_{i1} A$ *is finitely cogenerated injective for each* $i = 1, 2, \dots, m$,
 (ii) $e_{ij} A \cong J(e_{i,j-1} A)$ *for each* $i = 1, 2, \dots, m$ *and* $j = 2, 3, \dots, n(i)$.

(4) *For each* $f \in \mathrm{pi}(A)$*, one of the following conditions holds.*
 (i) fA *is finitely cogenerated injective,*
 (ii) $fA \cong J(eA)$ *for some* $e \in \mathrm{pi}(A)$ *and* fA *is co-hopfian.*

Proof. (1) $\Leftrightarrow$ (2). This is by [16, Theorem 2.2].

(3) $\Rightarrow$ (2). Let $\{e_{ij}\}$ be a basic set of (3). Clearly $e_{ij}A$ is small iff $j \neq 1$. Thus it suffices to show that $J(e_{i,n(i)}A)$ is singular for each i. Let $J = J(A)$ and let $f \in \mathrm{pi}(A)$ with $S(e_{i1}A) \cong T(fA)$. Then by using the assumption and Lemma 2.3, we see that $e_{i,n(i)} J S(A_A) = 0$. Since $S(A_A)$ is essential in A_A, this shows that $e_{i,n(i)} J$ is singular.

(2) $\Rightarrow$ (4). Let $\{e_i\} \cup \{f_j\}$ be a basic set of (2). By (1) $\Leftrightarrow$ (2) and [7, collolary on p. 435] A is a right QF-2 ring. Let $g \in \mathrm{pi}(A)$. If $gA \cong e_i A$ for some i, then gA is finitely cogenerated injective because A is right QF-2. If $gA \cong f_j A$ for some j, then by (2)(iii) $f_j A$ can be embedded in $e_i A$ for some i. Since A is right QF-2, all $J_0(e_i A), \dots, J_{t_i}(e_i A)$ are indecomposable projective. If $f_j A \ncong J_t(e_i A)$ for any $t = 0, 1, \dots, t_i$, then $f_j A$ must be embedded in $J_{t_i+1}(e_i A)$ and hence $f_j A$ is singular. Thus $S(f_j A) = f_j A \cdot S(A_A) = 0$, a contradiction. Therefore $f_j A \cong J_t(e_i A)$ for some t. Then $f_j A$ is quasi-injective and hence $f_j A$ is co-hopfian.

(4) $\Rightarrow$ (3). Assume (4). Since A is semiperfect and all indecomposable projective right A-modules are co-hopfian, we see that for each $f \in \mathrm{pi}(A)$ there exist $e \in \mathrm{pi}(A)$ and a non-negative integer t such that eA is finitely cogenerated injective and $fA \cong J_t(eA)$. By using this fact, we can easily verify (3). □

We call a semiperfect ring A satisfying the equivalent conditions of Lemma 2.4 a *right pseudo co-Harada ring* (abbreviated *right PCH ring*). Thus right co-Harada rings are precisely right PCH rings with ACC on right annihilators. We shall often use the condition (4) of Lemma 2.4 to show that a ring is right PCH. The main purpose of this section is to prove Theorem 2.17, which determines the structure of right PCH rings.

In this section, we do not assume rings to be one-sided artinian. Thus we cannot use [1, Theorem 31.3] (so called "Fuller's theorem" about injective pairs). Instead of the theorem, we often use the following lemma, which is also due to Fuller.

Lemma 2.5 [1, Lemma 31.2]. *Let X be an injective right A-module and let e be an idempotent of A such that $l_X(Ae) = 0$. Then the canonical natural transformation of functors*

$$\mathrm{Hom}_A(-, X) \to \mathrm{Hom}_{eAe}(- \otimes_A Ae, Xe)$$

is an isomorphism. In particular, Xe_{eAe} is injective.

The following two lemmas show that for a right PCH ring A and $e, f \in \mathrm{pi}(A)$ with $fA \cong J(eA)$, the ring $A' = (1-f)A(1-f)$ and the idempotent e of A' can reconstruct the ring A.

Lemma 2.6. *Let A be a basic semiperfect ring and let $e, f \in \mathrm{pi}(A)$ be orthogonal with $fA \cong J(eA)$. Let $f' = 1 - f$, let $A' = f'Af'$ and let $\widetilde{A} = A'_e$. Then there exists a ring homomorphism $\phi_e : \widetilde{A} \to A$ satisfying the following conditions.*

(1) $\mathrm{Ker}(\phi_e) = \left(\begin{smallmatrix} 0 & S(f'A_A)e \\ 0 & S(eA_A)e \end{smallmatrix}\right) \leqslant S(\widetilde{A}_{\widetilde{A}})\left(\begin{smallmatrix} 0 & 0 \\ 0 & e \end{smallmatrix}\right)$.
(2) *ϕ_e is surjective if and only if* $\mathrm{Ext}^1_A(T(eA), f'A) = 0$.
(3) *ϕ_e is injective if and only if* $\mathrm{Hom}_A(T(eA), f'A) = 0$.

Proof. Let $J = J(A)$. We note that

$$A = \begin{pmatrix} f'Af' & f'Af \\ fAf' & fAf \end{pmatrix} \quad \text{and} \quad \widetilde{A} = \begin{pmatrix} f'Af' & f'Ae \\ eJf' & eAe \end{pmatrix}.$$

Since $fA \cong eJ$, there exists a monomorphism $\kappa : fA \to eA$. Then by assumption, for any homomorphisms $\beta : eA \to f'A$, $\gamma : f'A \to eJ$ and $\delta : eA \to eA$, there exist homomorphisms $\beta' : fA \to f'A$, $\gamma' : f'A \to fA$ and $\delta' : fA \to fA$ making the following diagrams commutative:

$$\begin{tikzcd} eA \arrow[r, "\beta"] & f'A \\ & fA, \arrow[ul, "\kappa"] \arrow[u, "\beta'"'] \end{tikzcd} \qquad \begin{tikzcd} f'A \arrow[r, "\gamma"] \arrow[dr, "\gamma'"'] & eJ \\ & fA, \arrow[u, "\cong"'] \end{tikzcd} \qquad \begin{tikzcd} eA \arrow[r, "\delta"] & eA \\ fA \arrow[u, "\kappa"] \arrow[r, "\delta'"] & fA. \arrow[u, "\kappa"'] \end{tikzcd}$$

Using the matrix representations of A and $\widetilde{A}$, we define

$$\phi_e \begin{pmatrix} \alpha & \beta \\ \gamma & \beta \end{pmatrix} = \begin{pmatrix} \alpha & \beta' \\ \gamma' & \delta' \end{pmatrix}.$$

It is easy to see that ϕ_e is well-defined and is a ring homomorphism. The condition (1) follows immediately from the definition of ϕ_e and the fact that $l_A(J) = S(A_A)$. To check up the conditions (2) and (3), applying $\mathrm{Hom}_A(-, f'A)$ to the short exact sequence

$$0 \to fA \xrightarrow{\kappa} eA \to T(eA) \to 0,$$

we have the exact sequence

$$0 \to \operatorname{Hom}_A\bigl(T(eA), f'A\bigr) \to \operatorname{Hom}_A(eA, f'A) \to \operatorname{Hom}_A(fA, f'A)$$
$$\to \operatorname{Ext}_A^1\bigl(T(eA), f'A\bigr) \to 0.$$

Thus, since eA is a direct summand of $f'A$, it follows from the definition of ϕ_e that ϕ_e is injective iff the homomorphism $\operatorname{Hom}_A(eA, f'A) \to \operatorname{Hom}_A(fA, f'A)$ is a monomorphism iff $\operatorname{Hom}_A(T(eA), f'A) = 0$. Therefore we have (2). Similarly the condition (3) holds. □

Lemma 2.7. *Let A be a basic right PCH ring and let $e, f \in \operatorname{pi}(A)$ with $fA \cong J(eA)$. Then*

$$\operatorname{Ext}_A^1\bigl(T(eA), (1-f)A\bigr) = 0.$$

In particular, the ring homomorphism ϕ_e of Lemma 2.6 *is surjective.*

Proof. It suffices to show that $\operatorname{Ext}_A^1(T(eA), gA) = 0$ for each $g \in \operatorname{pi}(A)$ with $gA \not\cong fA$. Let $g \in \operatorname{pi}(A)$ with $gA \not\cong fA$ and let $\alpha : fA \to gA$ be a homomorphism. We claim that for any $h \in \operatorname{pi}(A)$, if the following commutative diagram with exact rows and columns is given, then β is not an epimorphism:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & fA \arrow[r] \arrow[d, "\alpha"] & eA \arrow[r] \arrow[d, "\beta"] & T(eA) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & gA \arrow[r] & hA \arrow[r] & X \arrow[r] & 0.
\end{tikzcd}$$

Indeed, if β is an epimorphism, then β is an isomorphism and hence so is γ. Thus α is an isomorphism, a contradiction. To show that α can be extended to $eA \to gA$, let $g_1, g_2, \ldots, g_n = g \in \operatorname{pi}(A)$ such that g_1A is injective and $g_iA \cong J(g_{i-1}A)$ for each i. Since g_1A is injective, there exists a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & fA \arrow[r] \arrow[d, "\alpha"] & eA \arrow[r] \arrow[d, "\beta"] & T(eA) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & gA \arrow[r] & g_1A \arrow[r] & X \arrow[r] & 0.
\end{tikzcd}$$

Then it follows from the claim that β factors through $gA \to g_1A$. □

Let $\widetilde{A} = A_e$ for a ring A and $e \in \operatorname{pi}(A)$. For $x \in A$ and $y \in eAe$, we define elements of $\widetilde{A}$ by

$$\tilde{x} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{y} = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}.$$

Lemma 2.8. *Let $\widetilde{A} = A_e$ for a ring A and $e \in \mathrm{pi}(A)$. Then for $f \in \mathrm{pi}(A)$, $\tilde{f}\widetilde{A}_{\widetilde{A}}$ is injective if and only if one of the following conditions holds.*

(1) *fA_A is injective and $fAe \cong \mathrm{Hom}_A(eJ(A), fA)$ canonically,*
(2) *fAe_{eAe} is injective and $fA \cong \mathrm{Hom}_{eAe}(Ae, fAe)$ canonically.*

In case (2) holds, fA_A is injective. Particularly, if $\tilde{f}\widetilde{A}_{\widetilde{A}}$ is injective, then fA_A is injective.

Proof. Regarding $\widetilde{A}$ as a ring extension of the product ring $A \times eAe$, we have the right $\widetilde{A}$-module $\mathrm{Hom}_{A\times eAe}(\widetilde{A}, X \times Y)$ for any right $(A \times eAe)$-module $X \times Y$. We note from [14, Corollary 2.2] that a right $\widetilde{A}$-module M is injective if and only if $M \cong \mathrm{Hom}_{A\times eAe}(\widetilde{A}, X \times Y)$ as right $\widetilde{A}$-modules for some injective right $(A \times eAe)$-module $X \times Y$.

($\Rightarrow$) Assume that $\tilde{f}\widetilde{A}_{\widetilde{A}}$ is injective. Then by the note above there exists an injective right $(A \times eAe)$-module $X \times Y$ such that $\tilde{f}\widetilde{A} \cong \mathrm{Hom}_{A\times eAe}(\widetilde{A}, X \times Y)$. Since $\tilde{f}\widetilde{A}$ is indecomposable, $X \times Y$ is also indecomposable. Thus either $Y = 0$ or $X = 0$. It is easy to see that (1) holds if $Y = 0$ and (2) holds if $X = 0$.

($\Leftarrow$) As is easily seen, $\tilde{f}\widetilde{A} \cong \mathrm{Hom}_{A\times eAe}(\widetilde{A}, fA \times 0)$ if (1) holds and $\tilde{f}\widetilde{A} \cong \mathrm{Hom}_{A\times eAe}(\widetilde{A}, 0 \times fAe)$ if (2) holds. Therefore $\tilde{f}\widetilde{A}_{\widetilde{A}}$ is injective by the note above.

To show the last assertion, assume that (2) holds. Then by (2) and the fact that ${}_A Ae$ is flat, the functors

$$\begin{aligned}\mathrm{Hom}_A(-, fA) &\cong \mathrm{Hom}_A\big(-, \mathrm{Hom}_{eAe}(Ae, fAe)\big)\\ &\cong \mathrm{Hom}_{eAe}(-\otimes_A Ae, fAe)\end{aligned}$$

are exact and hence fA_A is injective. □

Lemma 2.9. *Let $\widetilde{A} = A_e$ for a basic semiperfect ring A and $e \in \mathrm{pi}(A)$. For $f \in \mathrm{pi}(A)$, fA_A is finitely cogenerated injective if and only if so is $\tilde{f}\widetilde{A}_{\widetilde{A}}$.*

Proof. Since fA_A and $\tilde{f}\widetilde{A}_{\widetilde{A}}$ are indecomposable, we remark that they are finitely cogenerated injective if and only if they are injective with essential socle. Let $J = J(A)$ and let $e' = 1 - e$. Then

$$A = \begin{pmatrix} e'Ae' & e'Ae \\ eAe' & eAe \end{pmatrix} \quad \text{and} \quad \widetilde{A} = \begin{pmatrix} e'Ae' & e'Ae & e'Ae \\ eAe' & eAe & eAe \\ eJe' & eJe & eAe \end{pmatrix}.$$

Since $S(A_A) = l_A(J)$ and $S(\widetilde{A}_{\widetilde{A}}) = l_{\widetilde{A}}(J(\widetilde{A}))$, using

$$J(A) = \begin{pmatrix} e'Je' & e'Je \\ eJe' & eJe \end{pmatrix} \quad \text{and} \quad J\big(\widetilde{A}\big) = \begin{pmatrix} e'Je' & e'Je & e'Je \\ eJe' & eJe & eAe \\ eJe' & eJe & eJe \end{pmatrix},$$

we can represent the socles as the following forms

$$S(A_A) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \quad \text{and} \quad S\big(\widetilde{A}_{\widetilde{A}}\big) = \begin{pmatrix} M_{11} & 0 & M_{12} \\ M_{21} & 0 & M_{22} \\ M_{21} & 0 & M_{22} \end{pmatrix},$$

where $M_{11} = l_{e'Ae'}(e'J)$, $M_{12} = l_{e'Ae}(eJ)$, $M_{21} = l_{eAe'}(e'J)$, and $M_{22} = l_{eAe}(eJ)$.

($\Rightarrow$) Assume that fA_A is finitely cogenerated injective. Applying $\mathrm{Hom}_A(-, fA)$ to the short exact sequence

$$0 \to eJ \to eA \to T(eA) \to 0,$$

we have the exact sequence

$$0 \to \mathrm{Hom}_A\big(T(eA), fA\big) \to \mathrm{Hom}_A(eA, fA) \to \mathrm{Hom}_A(eJ, fA) \to 0.$$

Thus, since $fAe \cong \mathrm{Hom}_A(eA, fA)$, $fAe \cong \mathrm{Hom}_A(eJ, fA)$ canonically if and only if $\mathrm{Hom}_A(T(eA), fA) = 0$. If these equivalent conditions hold, then $\tilde{f}\widetilde{A}_{\widetilde{A}}$ is injective by Lemma 2.8(1). If the equivalent conditions do not hold, since $S(fA)$ is essential in fA and $S(fA) \cong T(eA)$, we have $l_{fA}(Ae) = 0$. Hence by Lemma 2.5 the condition (2) of Lemma 2.8 holds. Therefore $\tilde{f}\widetilde{A}_{\widetilde{A}}$ is injective by Lemma 2.8. It follows from the matrix representations of socles that $S(\tilde{f}\widetilde{A}_{\widetilde{A}})$ is essential in $\tilde{f}\widetilde{A}_{\widetilde{A}}$.

($\Leftarrow$) Assume that $\tilde{f}\widetilde{A}_{\widetilde{A}}$ is finitely cogenerated injective. Then by Lemma 2.8 fA_A is injective. It also follows from the matrix representations of socles that $S(fA_A)$ is essential in fA_A. $\square$

We now show the following proposition, which states the relationship of right PCH-ness between A and A_e.

Proposition 2.10. *For a ring A and $e \in \mathrm{pi}(A)$, A is a right PCH ring if and only if so is A_e.*

Proof. Let $J = J(A)$, $\widetilde{A} = A_e$, and $\widetilde{J} = J(\widetilde{A})$. We check up the condition of Lemma 2.4(4).

($\Rightarrow$) Assume that A is right PCH. It suffices to show that the condition of Lemma 2.4(4) holds for the elements of $\mathrm{pi}(\widetilde{A})$ of the forms $\tilde{f}$ ($f \in \mathrm{pi}(A)$) and $\hat{e}$.

First for $\hat{e}$, to show that $\hat{e}\widetilde{A}$ is co-hopfian, let $\alpha : \hat{e}\widetilde{A}_{\widetilde{A}} \to \hat{e}\widetilde{A}_{\widetilde{A}}$ be a monomorphism. Then α is given by the left multiplication map of $\hat{y}$ for some $y \in eAe$. The left multiplication maps $eJ \to eJ$ and $eAe \to eAe$ of y are monomorphisms. If $eJ \neq 0$, then the left multiplication map $eA \to eA$ of y is a monomorphism because eA has simple essential socle. Thus, since eA is co-hopfian, y is invertible. If $eJ = 0$, then eA is simple and eAe is a division ring. Hence $y \in eAe$ is invertible. Thus α is an isomorphism. We also have $\hat{e}\widetilde{A} \cong \tilde{e}\widetilde{J}$. Therefore $\hat{e}$ satisfies the condition of Lemma 2.4(4).

Next for $\tilde{f}(f \in \mathrm{pi}(A))$, as is easily seen, $\tilde{f}\widetilde{A}_{\widetilde{A}}$ is co-hopfian. If fA_A is injective, then by Lemma 2.9 $\tilde{f}\widetilde{A}_{\widetilde{A}}$ is finitely cogenerated injective. If fA_A is not injective, then $fA \cong gJ$ for some $g \in \mathrm{pi}(A)$. It is routine to verify that $\tilde{f}\widetilde{A} \cong \tilde{g}\widetilde{J}$ if $gA \ncong eA$ and $\tilde{f}\widetilde{A} \cong \hat{e}\widetilde{J}$ if $gA \cong eA$. Thus $\tilde{f}$ also satisfies the condition of Lemma 2.4(4). Therefore $\widetilde{A}$ is right PCH.

($\Leftarrow$). Assume that $\widetilde{A}$ is right PCH. Let $f \in \mathrm{pi}(A)$. If $\tilde{f}\widetilde{A}_{\widetilde{A}}$ is injective, then by Lemma 2.8 fA_A is finitely cogenerated injective. If $\tilde{f}\widetilde{A}_{\widetilde{A}}$ is not injective, there are two cases. In case $\tilde{f}\widetilde{A} \cong \hat{e}\widetilde{J}$, we have $fA \cong eJ$. In case $\tilde{f}\widetilde{A} = \tilde{g}\widetilde{J}$ for some $g \in \mathrm{pi}(A)$, we have $fA \cong gJ$. Since $\tilde{f}\widetilde{A}_{\widetilde{A}}$ is co-hopfian, it is easy to see that fA is co-hopfian. Therefore A is right PCH by Lemma 2.4. □

To prove Theorem 2.17, which is the main purpose of this section, we need to show a key result Proposition 2.15. The proposition is divided into three cases. The first case is immediate from Proposition 2.10 as follows.

Lemma 2.11. *Let A be a basic right PCH ring and let $e, f \in \mathrm{pi}(A)$ with $fA \cong J(eA)$. If $S(A_A)e = 0$, then $(1-f)A(1-f)$ is a right PCH ring.*

Proof. Let $A' = (1-f)A(1-f)$ and let $\widetilde{A} = A'_e$. Then by Lemmas 2.6 and 2.7 the ring homomorphism $\phi_e : \widetilde{A} \to A$ is an isomorphism. Thus by Proposition 2.10 A' is right PCH. □

We prove the second case of Proposition 2.15 as follows.

Lemma 2.12. *Let A be a basic right PCH ring and let $e, f \in \mathrm{pi}(A)$ with $fA \cong J(eA)$. If $S(A_A)f = 0$, then $(1-f)A(1-f)$ is a right PCH ring.*

Proof. Let $J = J(A)$, $f' = 1 - f$ and $A' = f'Af'$. We check up the condition of Lemma 2.4(4). Let $g \in \mathrm{pi}(A')$, i.e., $g \in \mathrm{pi}(A)$ with $gA \ncong fA$.

We first show that $gA'_{A'}$ is co-hopfian. Let $\alpha : gA'_{A'} \to gA'_{A'}$ be a monomorphism. Then α is given by the left multiplication map of an element gag of $gA'g = gAg$. If $(gag)S(gA) = 0$, then $\alpha(S(gA)f') = 0$ and hence $S(gA)f' = 0$. Thus by the assumption $S(A_A)f = 0$, we have $S(gA) = 0$, a contradiction. Therefore the left multiplication map $gA_A \to gA_A$ of gag is a monomorphism. Since gA_A is co-hopfian, gag is invertible and hence α is an isomorphism.

If gA is finitely cogenerated injective, then $l_{gA}(Af') = 0$ and $gA'f'_{f'Af'} = gA'_{A'}$ is injective by Lemma 2.5. Since $S(gA)$ is essential in gA, $S(gA'_{A'})$ is essential in $gA'_{A'}$ and $gA'_{A'}$ is finitely cogenerated. If gA is not injective, then there exists $h \in \mathrm{pi}(A)$ such that $gA \cong hJ$. In case $hA \ncong fA$, we have $gA'_{A'} \cong J(hA'_{A'})$. In case $hA \cong fA$, since fA is not injective, there exists $k \in \mathrm{pi}(A)$ such that

$fA \cong kJ$. Then it is easy to see that $gA'_{A'} \cong J(kA'_{A'})$. Therefore by Lemma 2.4 A' is right PCH. □

To prove the third case of Proposition 2.15, we need the following lemma.

Lemma 2.13. *Let A be a basic right PCH ring and let $e, f, g \in \mathrm{pi}(A)$ such that*

(a) $fA \cong J(eA)$,
(b) *gA is injective and $S(gA) \cong T(fA)$.*

Let $f' = 1 - f$ and $e' = 1 - e'$. Then

(1) *There exists a surjective ring homomorphism $\psi : f'Af' \to e'Ae'$ such that*
 (i) $\mathrm{Ker}(\psi) = S(f'A_A)e \leqslant J(f'Af')$,
 (ii) $(gAf')(\mathrm{Ker}(\psi)) = 0$.
(2) *There exists an isomorphism $\lambda : {}_{gAg}gAf' \to {}_{gAg}gAe'$ such that $\lambda(\beta\alpha) = \lambda(\beta)\psi(\alpha)$ for each $\alpha \in f'Af'$ and $\beta \in gAf'$.*

Furthermore, let $h_1 = h, h_2, \ldots, h_n \in \mathrm{pi}(A)$ such that

(c) *hA is injective and $S(hA) \cong T(eA)$,*
(d) *$h_iA \cong J(h_{i-1}A)$ for each $i = 2, 3, \ldots, n$ and $J(h_nA)$ is not projective.*

Then

(3) *$hAf'_{f'Af'}$ is injective.*
(4) $gAf'_{f'Af'} \cong h_nAf'_{f'Af'}$.

Proof. (1) Let $J = J(A)$ and let $k = 1 - e - f$. Then k is an idempotent of A, $e + k = f'$ and $f + k = e'$. Since $fA \cong eJ < eA$, taking the direct sum of the monomorphism $fA \to eA$ and the identity map $kA \to kA$, we have a monomorphism $\kappa : e'A \to f'A$. For any homomorphism $\alpha : f'A \to f'A$, since A is basic, there exists a unique homomorphism $\alpha : e'A \to e'A$ making the following square commutative:

$$\begin{array}{ccc} f'A & \xrightarrow{\alpha} & f'A \\ \uparrow{\scriptstyle\kappa} & & \uparrow{\scriptstyle\kappa} \\ e'A & \xrightarrow{\alpha'} & e'A. \end{array}$$

We then define $\psi(\alpha) = \alpha'$. It follows from Lemma 2.7 that ψ is a surjective ring homomorphism. It is routine to check ψ satisfies the conditions (i) and (ii).

(2) For any homomorphism $\beta : f'A \to gA$ there exists a unique homomorphism $\beta' : e'A \to gA$ making the following triangle commutative:

$$\begin{tikzcd} f'A \arrow[r, "\beta"] & gA \\ e'A \arrow[u, "\kappa"] \arrow[ur, "\beta'"'] & \end{tikzcd} .$$

We then define $\lambda(\beta) = \beta'$. Since gA_A is injective and $S(gA) \cong T(fA) \ncong T(eA)$, $\lambda : {}_{gAg}gAf' \to {}_{gAg}gAe'$ is an isomorphism. By the definitions of ψ and λ, for each $\alpha \in f'Af'$ and $\beta \in gAf'$, the following diagram is commutative:

$$\begin{tikzcd} f'A \arrow[r, "\alpha"] & f'A \arrow[r, "\beta"] & gA \\ e'A \arrow[u, "\kappa"] \arrow[r, "\psi(\alpha)"] & e'A \arrow[u, "\kappa"] \arrow[ur, "\lambda(\beta)"'] & \end{tikzcd} .$$

This shows that $\lambda(\beta\alpha) = \lambda(\beta)\psi(\alpha)$.

(3) Since $S(hA) \cong T(eA)$, we have $l_{hA}(Af') = 0$. Hence by Lemma 2.5 $hAf'_{f'Af'}$ is injective.

(4) Since gA_A has simple essential socle $S(gA_A) \cong T(fA_A)$, it follows from Lemma 2.1 that $gAe'_{e'Ae'}$ has simple essential socle $S(gAe'_{e'Ae'}) = S(gA_A)f = S(gA_A)$. Thus from (1), (2), and $\psi(e) = f$, we have that $gAf'_{f'Af'}$ has simple essential socle $S(gAf'_{f'Af'}) = S(gAf'_{f'Af'})e$. Therefore $S(gAf'_{f'Af'}) \cong T(eAf'_{f'Af'})$. On the other hand, since hA_A has simple essential socle $S(hA_A) \cong T(eA_A)$, we can show that $hAf'_{f'Af'}$ has simple essential socle $S(hAf'_{f'Af'}) \cong T(eAf'_{f'Af'})$. Thus we have $S(gAf'_{f'Af'}) \cong S(hAf'_{f'Af'})$.

Therefore, since $hAf'_{f'Af'}$ is injective by (3), there exists an $f'Af'$-homomorphism $\alpha : gAf' \to hAf'$ making the following square commutative:

$$\begin{tikzcd} S(gAf'_{f'Af'}) \arrow[r, "\leqslant"] \arrow[d, "\cong"] & gAf' \arrow[d, "\alpha"] \\ S(hAf'_{f'Af'}) \arrow[r, "\leqslant"] & hAf'. \end{tikzcd}$$

Then α is a monomorphism because the inclusion of the upper row is essential. By $S(hA) \cong T(eA)$ we have $l_{hA}(Af') = 0$. Therefore, since by Lemma 2.5 $\mathrm{Hom}_A(gA, hA) \cong \mathrm{Hom}_{f'Af'}(gAf', hAf')$, there exists an A-homomorphism $\beta : gA \to hA$ such that $\beta|_{gAf'} = \alpha$. By the assumption (d) there exists a monomorphism $\gamma : h_nJ \to hA$. Thus, since $S(gA) \ncong S(hA)$, there exists a homomorphism $\delta : gA \to h_nJ$ such that $\beta = \gamma\delta$. Let $\epsilon = \delta|_{gAf'} : gAf'_{f'Af'} \to h_nJf'_{f'Af'}$. Then ϵ is an essential monomorphism because ϵ is induced from α.

Since $T(fA) \cong S(gA)$, we have $l_{gA}(Ae') = 0$. Thus $gAe'_{e'Ae'}$ is injective by Lemma 2.5. By using (1), (2) and the surjective ring homomorphism $\psi : f'Af' \to$

$e'Ae'$, we can regard gAf' as an injective right $e'Ae'$-module. It follows from Lemma 2.3 and (1) that $(h_nJf')(\mathrm{Ker}(\psi)) = 0$. Therefore h_nJf' becomes a right $e'Ae'$-module via ψ. Then we can regard $\epsilon : gAf' \to h_nJf'$ as an essential right $e'Ae'$-monomorphism. Thus ϵ must be an isomorphism and hence we have $gAf'_{f'Af'} \cong h_nJf'_{f'Af'}$. □

Now we can prove the third case of Proposition 2.15 as follows.

Lemma 2.14. *Let A be a basic right PCH ring and let $e, f \in \mathrm{pi}(A)$ with $fA \cong J(eA)$. If $S(A_A)e \neq 0$ and $S(A_A)f \neq 0$, then $(1-f)A(1-f)$ is a right PCH ring.*

Proof. Let $J = J(A)$, let $f' = 1 - f$ and let $A' = f'Af'$. Let $g \in \mathrm{pi}(A')$, i.e., $g \in \mathrm{pi}(A)$ with $gA \ncong fA$. We check up the condition of Lemma 2.4(4).

In case gA is finitely cogenerated injective with $S(gA) \ncong T(fA)$, by a similar way of the proof of Lemma 2.12, we see that $gA'_{A'}$ is finitely cogenerated injective.

In case gA is finitely cogenerated injective with $S(gA) \cong T(fA)$, since $S(A_A)e \neq 0$, there exist $h_1 = h, h_2, \dots, h_n \in \mathrm{pi}(A)$ such that $S(hA) \cong T(eA)$, $h_iA \cong h_{i-1}J$ for each i and h_nJ is not projective. Then $gAf'_{f'Af'} \cong h_nJf'_{f'Af'}$ by Lemma 2.13. If $h_nA \cong fA$, then we have $gA'_{A'} \cong J(eA'_{A'})$ and $e \in \mathrm{pi}(A')$. If $h_nA \ncong fA$, then we have $gA'_{A'} \cong J(h_nA'_{A'})$ and $h_n \in \mathrm{pi}(A')$.

In case gA is not injective, $gA \cong hJ$ for some $h \in \mathrm{pi}(A)$. If $hA \ncong fA$, then we have $gA'_{A'} \cong J(hA'_{A'})$ and $h \in \mathrm{pi}(A')$. If $hA \cong fA$, then we have $gA'_{A'} \cong J(eA'_{A'})$ and $e \in \mathrm{pi}(A')$. Thus we see that $gA' \cong J_t(kA')$ for some $k \in \mathrm{pi}(A')$ with kA' injective and some positive integer t. Therefore gA' is co-hopfian. □

We now obtain the following proposition, which is a key result in the proof of Theorem 2.17.

Proposition 2.15. *Let A be a basic right PCH ring and let $f \in \mathrm{pi}(A)$ with fA non-injective. Then $(1-f)A(1-f)$ is a right PCH ring.*

Proof. Since fA is not injective, there exists $e \in \mathrm{pi}(A)$ such that $fA \cong J(eA)$. For e and f, there are three cases (1) $S(A_A)e = 0$, (2) $S(A_A)f = 0$, and (3) $S(A_A)e \neq 0$ and $S(A_A)f \neq 0$. These cases are proved in Lemmas 2.11, 2.12 and 2.14, respectively. □

Remark 2.16. For a right PCH ring A, let denote $m(A)$ the number of isomorphism classes of indecomposable projective injective right A-modules. Let $e, f \in \mathrm{pi}(A)$ with $fA \cong J(eA)$ and let $A' = (1-f)A(1-f)$. Then the proofs of Lemmas 2.11, 2.12, and 2.14 show that $m(A') = m(A)$ if either $S(A_A)e = 0$ or $S(A_A)f = 0$ and $m(A') = m(A) - 1$ if $S(A_A)e \neq 0$ and $S(A_A)f \neq 0$.

We can now obtain the main purpose of this section. The theorem shows a relationship between right PCH rings and right PF rings, and determines the structure of right PCH rings.

Theorem 2.17 (cf. [18, Theorem 2]). *Let A be a basic right PCH ring. Then there exist primitive idempotents $e_1, e_2, \ldots, e_l$ of A and orthogonal primitive idempotents $f_1, f_2, \ldots, f_l$ of A satisfying the following conditions:*

(1) *$A_0, \ldots, A_{l-1}$ are right PCH rings and A_l is a right PF ring,*
(2) *for each $i = 1, 2, \ldots, l$, there exists a surjective ring homomorphism $\phi_i : \widetilde{A}_i \to A_{i-1}$ such that* $\mathrm{Ker}(\phi_i) \leqslant S(\widetilde{A}_{i_{\widetilde{A}_i}})$,

where $A_0 = A$, $A_i = (1 - \sum_{j=1}^{i} f_j)A(1 - \sum_{j=1}^{i} f_j)$ and $\widetilde{A}_i = A_{i_{e_i}}$.

Proof. We prove the statement by induction on the number n of the idempotents of a basic set of primitive idempotents for A.

In case $n = 1$, A is a local right PF ring and there is nothing to prove. In case $n > 1$, if A is right PF, then there is also nothing to prove. Thus we assume that A is not right PF. Then there exists $e_1, f_1 \in \mathrm{pi}(A)$ such that $f_1 A \cong J(e_1 A)$. Let $A_1 = (1 - f_1)A(1 - f_1)$. By Proposition 2.15 A_1 is right PCH. Hence by the hypothesis of induction, there exist primitive idempotents $e_2, \ldots, e_l$ of A_1 and orthogonal primitive idempotents $f_2, \ldots, f_l$ of A_1 satisfying the conditions of Theorem 2.17. By Lemmas 2.6 and 2.7 there exists a surjective ring homomorphism $\phi_1 : \widetilde{A}_1 \to A_0$ such that $\mathrm{Ker}(\phi_1) \leqslant S(\widetilde{A}_{1_{\widetilde{A}_1}})$. Therefore we obtain the required idempotents $e_1, e_2, \ldots, e_l, f_1, f_2, \ldots, f_l$ of A. $\square$

Example 2.18. We illustrate Theorem 2.17 by a simple example.

Let K be a field and let Q be the quiver $1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3$. Let $A = KQ/\langle\alpha\beta\rangle$ be the factor K-algebra of the path algebra KQ and let g_i be the primitive idempotent of A corresponding to the vertex i for $i = 1, 2, 3$. Then it is routine to see that $g_1 A$ and $g_2 A$ are injective and $g_3 A \cong J(g_2 A)$. Therefore A is a right co-Harada ring. Indeed, A is a serial ring with admissible sequence (2,2,1).

Let $A_1 = (1 - g_3)A(1 - g_3)$. Clearly A_1 has the form KQ' for the quiver $Q' : 1 \xleftarrow{\alpha} 2$. It is easy to see that $\widetilde{A}_1 = A_{1_{g_2}}$ has the form KQ. Thus $\widetilde{A}_1/\langle\alpha\beta\rangle \cong A$. Moreover, let $A_2 = (1 - g_2)A_1(1 - g_2)$. Then $A_2 = K$ and $A_1 = \widetilde{A}_2 = A_{2_{g_1}}$. Therefore the idempotents of Theorem 2.17 for A are $e_1 = g_2$, $e_2 = g_1$, $f_1 = g_3$, $f_2 = g_2$ and A can be constructed from a QF ring (indeed a field) $A_2 = K$ as this way.

3. Applications to dualities

In the final section we mainly apply the results of preceding sections to dualities of PCH rings and co-Harada rings. Especially, we obtain Theorem 3.2, the main purpose of the paper.

We recall that a semiperfect ring A is a right co-Harada if A satisfies the condition $(*)^*$ and ACC on *right* annihilators. As an application of Theorem 2.17, we have

Proposition 3.1. *A ring A is a right co-Harada ring if and only if A is a right PCH ring with ACC on left annihilators.*

Proof. In view of Lemma 2.4, it suffices to show that every right PCH ring A with ACC on left (respectively right) annihilators is left and right artinian. If a right PCH ring A satisfies ACC on left (respectively right) annihilators, then the right PF ring A_l of Theorem 2.17 satisfies ACC on left (respectively right) annihilators. Then by [5, 24.22] A_l is QF. Therefore again by Theorem 2.17 A is left and right artinian. □

As we used in the proof of Lemma 1.11, the notion of linearly compact modules is closely related to the theory of Morita duality. A ring A is said to be a *right* (respectively *left*) *linearly compact ring* if the regular module A_A (respectively ${}_AA$) is linearly compact. As is well known, right artinian rings are right linearly compact rings. Therefore right co-Harada rings are right linearly compact right PCH rings.

We now prove the main purpose of the paper.

Theorem 3.2. *Every right linearly compact right PCH ring has an almost self-duality. In particular, every right co-Harada ring has an almost self-duality.*

Proof. Let A be a right linearly compact right PCH ring. We may assume that A is basic. We use Theorem 2.17 and its notation. Since A is right linearly compact, the right PF ring A_l is also a right linearly compact ring by [21, Corollary 3.18]. Thus A_l has a self-duality by [21, Theorem 4.3] and A_l is a PF ring. In general, if a ring B has an almost self-duality, then so does B_f for each $f \in \mathrm{pi}(B)$ by Proposition 1.14. Therefore, in view of Theorem 2.17, it suffices to show that if a right PCH ring B has an almost self-duality, then so does B/K for any ideal K of B with $K \leqslant S(B_B)$. Let $\{f_1, f_2, \ldots, f_n\}$ be a complete set of orthogonal primitive idempotents for B. For any ideal K of B with $K \leqslant S(B_B)$, let $I = \{i \mid f_iK \neq 0\}$ and $f = \sum_{i\in I} f_i$. Then, since B is right QF-2, $f_iK = S(f_iB_B)$ for $i \in I$. Therefore we have

$$K = \sum_{i\in I} f_iK = \sum_{i\in I} S(f_iB_B) = \left(\sum_{i\in I} f_i\right)S(B_B) = fS(B_B) = BfS(B_B).$$

Hence by Lemma 1.9 B/K has an almost self-duality. □

Example 3.3. In [10] the author constructed right co-Harada rings without self-duality. We now present one of the examples [10, Example 3.1]. Let $A_1, A_2, \dots,$ A_5 be the artinian rings in Example 1.3 and let ${}_{A_2}U_{1A_1}, {}_{A_3}U_{2A_2}, \dots, {}_{A_1}U_{5A_5}$ be bimodules that define a Morita duality. Let $B = A_1 \times A_2 \times \cdots \times A_5$ be the product ring and let $V = U_1 \oplus U_2 \oplus \cdots \oplus U_5$ be the direct sum of additive groups. Then V becomes a (B, B)-bimodule that defines a self-duality and the trivial extension $R = B \ltimes V$ is a QF ring without weakly symmetric self-duality. (See the proof of Theorem 1.6.) Let $e_i \in R$ be the idempotent corresponding to the idempotent of B of the ith projection and let $R_i = R_{e_i}$ for $i = 1, 2, \dots, 5$. By Lemma 1.13 each R_i is right Morita dual to R_{i+1} for $1 \leqslant i \leqslant 4$ and R_5 is right Morita dual to R_1. Then each of R_i is a right co-Harada ring with almost self-duality and by [10, Corollary 3.1] none of R_i has a self-duality. From the constructions of A_i, all R_i have twelve isomorphism classes of simple modules.

From Theorems 3.2 and 1.4, we have the following corollary.

Corollary 3.4. *For every right linearly compact right PCH ring* (*respectively right co-Harada ring*) A, *there exist a PF ring* (*respectively QF ring*) R *and an idempotent* e *of* R *such that* $A \cong eRe$.

We recall that a PF ring A is *weakly symmetric* if $S(eA) \cong T(eA)$ for each $e \in \mathrm{pi}(A)$. For the existence of weakly symmetric self-duality, we have

Theorem 3.5 (cf. [9, Theorem 5.1]). *A right linearly compact right PCH ring* A *has a weakly symmetric self-duality if and only if the PF ring* A_l *of Theorem* 2.17 *has a weakly symmetric self-duality.*

Proof. By using a similar way of the proof of Theorem 3.2, this follows from Lemmas 1.2(3) and 1.9(3), Proposition 1.14(3), and Theorem 2.17. □

As we mentioned before, Oshiro [18] proved the interesting result that a ring A is right co-Harada if and only if A is left Harada. We extend this result to right linearly compact right PCH rings in Theorem 3.7. To do this, we introduce the class of PH rings.

Dualizing the condition of Lemma 2.4(3), we say that a semiperfect ring A is a *left pseudo Harada ring* (abbreviated *left PH ring)* if there exists a complete irredundant set $\{E_{ij} \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n(i)\}$ of isomorphism classes of finitely cogenerated indecomposable injective left A-modules satisfying

(i) E_{i1} is finitely generated projective for each $i = 1, 2, \dots, m$,
(ii) $E_{ij} \cong E_{i,j-1}/S(E_{i,j-i})$ for each $i = 1, 2, \dots, m$ and $j = 2, \dots, n(i)$.

Left Harada rings are precisely left artinian left PH rings. (See [2, Theorem B].)

To prove Theorem 3.7, we need to recall the definition of Nakayama pair. Let A be a semiperfect ring and let $e, f \in \mathrm{pi}(A)$. Following Xue [22], we say that the pair (eA, Af) is a *Nakayama pair* if $S(eA) \cong T(fA)$, $S(Af) \cong T(Ae)$ and both eA_A and ${}_AAf$ have essential socles.

Lemma 3.6. *Let A be a basic right PCH ring and let $e_1 = e, e_2, \ldots, e_n, f \in \mathrm{pi}(A)$ such that*

(a) *eA is injective and $S(eA) \cong T(fA)$,*
(b) *$e_iA \cong J(e_{i-1}A)$ for each $i = 2, 3, \ldots, m$.*

For each $k = 0, 1, \ldots, n-1$, the pair $(\overline{e_{k+1}}\overline{A}_{\overline{A}}, {}_{\overline{A}}\overline{A}\,\bar{f})$ is a Nakayama pair, where $\overline{A} = A/S_k(Af)$.

Proof. Let $J = J(A)$. For the sake of convenience, set $e_0 = 0$. Then by Lemma 2.3 $S_j(Af) = \sum_{i=0}^{j} S(e_iA)$ for $0 \leqslant j \leqslant n$. In particular, we have $S_{k+1}(Af) = S(e_{k+1}A) + S_k(Af)$. Since $S_k(Af) < S_n(Af)$, we have $S_k(Af) \leqslant J$. Thus we note that $S(X_{\overline{A}}) = S(X_A)$ for any right $\overline{A}$-module X.

We first verify that $\overline{e_{k+1}}\overline{A}$ has simple essential socle $S(\overline{e_{k+1}}\overline{A}_{\overline{A}}) \cong T(\bar{f}\overline{A}_{\overline{A}})$. Since

$$\overline{e_{k+1}}\overline{A} = \frac{e_{k+1}A}{e_{k+1}S_k(Af)} = e_{k+1}A,$$

by the note above $\overline{e_{k+1}}\overline{A}_{\overline{A}}$ has essential socle. Furthermore, since

$$S(e_{k+1}A) \cong S(eA) \cong T(fA),$$

we have $S(\overline{e_{k+1}}\overline{A}_{\overline{A}}) \cong T(\bar{f}\overline{A}_{\overline{A}})$.

We next verify that ${}_{\overline{A}}\overline{A}\bar{f}$ has simple socle $S({}_{\overline{A}}\overline{A}\bar{f}) \cong T({}_{\overline{A}}\overline{A}\overline{e_{k+1}})$. Let $\bar{\alpha}, \bar{\beta}$ be nonzero elements of

$$S\big({}_{\overline{A}}\overline{A}\bar{f}\big) = \frac{S_{k+1}(Af)}{S_k(Af)}.$$

Since

$$S_{k+1}(Af) = S(e_{k+1}A) + S_k(Af),$$

we may assume that $\alpha, \beta \in S(e_{k+1}A)$. Regarding α, β as homomorphisms $A_A \to S(e_{k+1}A_A)$, we obtain a homomorphism $\gamma : S(e_{k+1}A) \to S(e_{k+1}A)$ such that $\beta = \gamma\alpha$. Since A is right PCH, there exists an extension $\delta : e_{k+1}A \to e_{k+1}A$ of γ. Then we have $\beta = \delta\alpha$ and this shows that $S({}_{\overline{A}}\overline{A}\bar{f})$ is simple. Also again by $S_{k+1}(Af) = S(e_{k+1}A) + S_k(Af)$ we see that $\overline{e_{k+1}}S({}_{\overline{A}}\overline{A}\,\bar{f}) \neq 0$. Therefore $S({}_{\overline{A}}\overline{A}\bar{f}) \cong T({}_{\overline{A}}\overline{A}\,\overline{e_{k+1}})$.

Finally, we verify that ${}_{\overline{A}}\overline{A}\bar{f}$ has essential socle. Let $\bar{\alpha}$ be a nonzero element of $\overline{A}\bar{f} = Af/S_k(Af)$, i.e., $\alpha \in Af$ and $\alpha \notin S_k(Af)$. We may assume $\alpha \in gAf$ for some $g \in \mathrm{pi}(A)$ and regard α as a homomorphism $fA \to gA$. It follows from Lemma 2.2 that there exists $\beta \in \mathrm{Hom}_A(gA, e_{k+1}A)$ such that $\mathrm{Im}(\beta\alpha) = S(e_{k+1}A)$. Then

$$0 \neq \bar{\beta}\bar{\alpha} \in \frac{S_{k+1}(Af)}{S_k(Af)} = S\big({}_{\overline{A}}\overline{A}\bar{f}\big).$$

Therefore $S({}_{\overline{A}}\overline{A}\bar{f})$ is essential in ${}_{\overline{A}}\overline{A}\bar{f}$. □

Although the proof of the implication ($\Rightarrow$) of the following theorem is a modification of that of [16, Theorem 3.7], the implication ($\Leftarrow$) is an easy consequence from the existence of almost self-duality for right linearly compact right PCH rings.

Theorem 3.7 (cf. [16, Theorems 3.7 and 5.5]). *A ring A is a right linearly compact right PCH ring if and only if A is a left linearly compact left PH ring.*

Proof. ($\Rightarrow$). Let A be a right linearly compact right PCH ring. We note that A is left linearly compact by Theorem 3.2. Let $e_1 = e, e_2, \ldots, e_n, f \in \mathrm{pi}(A)$ such that

(a) eA is injective and $S(eA) \cong T(fA)$,
(b) $e_iA \cong J(e_{i-1}A)$ for each $i = 2, 3, \ldots, n$ and $J(e_nA)$ is not projective.

By the definition of left PH rings, it suffices to show that $Af/S_k(Af)$ is isomorphic to an injective hull of $T(Ae_{k+1})$ for each $k = 0, 1, \ldots, n-1$.

For $k = 0, 1, \ldots, n-1$, let $\overline{A} = A/S_k(Af)$. Then $\overline{A}$ is left and right linearly compact because so is A. Thus, since $(\overline{e_{k+1}}\overline{A}_{\overline{A}}, {}_{\overline{A}}\overline{A}\bar{f})$ is a Nakayama pair by Lemma 3.6, ${}_{\overline{A}}\overline{A}\bar{f} = {}_{\overline{A}}Af/S_k(Af)$ is injective by [22, Theorem 7] and ${}_A\overline{A}\bar{f}$ has simple essential socle $S({}_A\overline{A}\bar{f}) \cong T({}_AAe_{k+1})$. Therefore it suffices to show that ${}_A\overline{A}\bar{f}$ is injective. We first observe that ${}_A\overline{A}\bar{f}$ is Ag-injective for each $g \in \mathrm{pi}(A)$ with $Ag \ncong Af$. Indeed, since $S_k(Af)g = 0$, Ag becomes a left $\overline{A}$-module. Hence by the injectivity of ${}_{\overline{A}}\overline{A}\bar{f}$, ${}_A\overline{A}\bar{f}$ is Ag-injective. Thus from [1, Proposition 16.12(2)] it remains to show that ${}_A\overline{A}\bar{f}$ is Af-injective. Let K be a left A-submodule of Af and let $\alpha : {}_AK \to {}_A\overline{A}\bar{f}$ be a nonzero homomorphism. By Lemma 3.6 $S_k(Af)$ is a uniserial module with the composition series

$$S_k(Af) > S_{k-1}(Af) > \cdots > S(Af) > 0.$$

Again by Lemma 3.6 it must hold that $K = S_i(Af)$ for some $0 \leqslant i \leqslant k$ or $K > S_k(Af)$. If $K = S_i(Af)$, then $e_{k+1}((K)\alpha) = (e_{k+1}K)\alpha = 0$. This contradicts that $S(Af/S_k(Af)) = S_{k+1}(Af)/S_k(Af)$ is essential in $Af/S_k(Af)$. Therefore we must have $K > S_k(Af)$. In this case, we also have $(S_k(Af))\alpha = 0$. Thus α

induces a left A-homomorphism $\bar{\alpha}: K/S_k(Af) \to \overline{A}\bar{f}$. Since ${}_{\overline{A}}\overline{A}\bar{f}$ is injective, $\bar{\alpha}$ can be extended to a left A-homomorphism $\beta: A/S_k(Af) \to \overline{A}\bar{f}$. Composing β and the canonical epimorphism $A \to A/S_k(Af)$, we have an extension ${}_AAf \to {}_A\overline{A}\bar{f}$ of α. Therefore ${}_A\overline{A}\bar{f}$ is injective.

($\Leftarrow$). Let A be a left linearly compact left PH ring. Then by the definition of left PH rings, a minimal injective cogenerator for the category of left A-modules is linearly compact. Hence by [21, Theorem 4.3] A has a left Morita duality. Let ${}_AU_B$ be a bimodule that defines a Morita duality. Since A is left PH, there exists a basic set $\{f_{ij} \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n(i)\}$ of primitive idempotents of B such that

(i) ${}_AUf_{i1}$ is finitely generated projective for each $i = 1, 2, \ldots, m$,
(ii) ${}_AUf_{ij} \cong {}_A(Uf_{i,j-1}/S(Uf_{i,j-1}))$ for each $i = 1, 2, \ldots, m$ and $j = 2, 3, \ldots, n(i)$.

Taking U-dual of these modules, we see from Lemma 2.4 that B is a right linearly compact right PCH ring. Thus by Theorem 3.2 B has an almost self-duality and hence A also has an almost self-duality. By the implication ($\Rightarrow$), B is a left linearly compact left PH ring. Let $A_1 = A, A_2 = B, \ldots, A_t, A_{t+1} = A$ be rings such that each A_i is left Morita dual to A_{i+1}. Then by the argument above $A_{t+1} = A$ is a right linearly compact right PCH ring. $\square$

Remark 3.8. (1) By Theorem 3.7 and its proof, in case ${}_BU_A$ defines a Morita duality, A is left linearly compact left PH (respectively right linearly compact right PCH) if and only if B is left linearly compact left PH (respectively right linearly compact right PCH).

(2) Without the linearly compactness, neither ($\Rightarrow$) nor ($\Leftarrow$) of Theorem 3.7 holds. Indeed, there exists a local left PF ring A that is not right PF (see [3].) The ring A is left PCH but not right PH and A is left PH but not right PCH.

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